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Weighted norm inequalities for multilinear Fourier multipliers with mixed norm

Abstract: Weighted norm inequalities for multilinear Fourier multipliers satisfying Sobolev regularity with mixed norm are discussed. Our result can be understood as a generalization of the result by Fujita and Tomita [4] by using the L^r-based Sobolev space, $1 < r \le 2$ with mixed norm.

1 Introduction and main result

Let $n \in \mathbb{N}$ and let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing smooth functions. Also, let N be a natural number, $N \ge 2$ and let $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi,$$

where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \dots d\xi_N$. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^d)$ satisfying

$$\operatorname{supp} \Psi \subset \left\{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\right\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1.1}$$

We set

$$m_j(\xi_1, \dots, \xi_N) = m(2^j\xi_1, \dots, 2^j\xi_N)\Psi(\xi_1, \dots, \xi_N), \quad j \in \mathbb{Z},$$

where Ψ is as in (1.1) with $d = Nn$. By $\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)}$, we denote the smallest constant C satisfying

$$\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \le C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}, \quad f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n).$$

Function spaces will be defined in Section 2.

In the unweighted case, Tomita [13] proved a Hörmander type multiplier theorem for multilinear operators, namely, if $s > Nn/2$, then

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^2_s(\mathbb{R}^{Nn})}$$

for $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$. Here $H^2_s(\mathbb{R}^{Nn})$ is the L^2-based Sobolev space of usual type. Grafakos and Si [7] extended this result to the case $p \le 1$ by using the L^r-based Sobolev space, $1 < r \le 2$. For further results in this direction, see [6, 10, 11, 5]. Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. In the weighted case, Fujita and Tomita [4] proved that if $n/2 < s_i \le n$, $p_i > n/s_i$ and $w_i \in A_{p_i s_i/n}$ for all $i = 1, \dots, N$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^2_{\vec{s}}(\mathbb{R}^{Nn})}, \tag{1.2}$$

where $w = w_1^{p/p_1} \dots w_N^{p/p_N}$ and $H^2_{\vec{s}}(\mathbb{R}^{Nn})$ denotes the L^2-based Sobolev space of product type. This result can also be obtained from another approach of [8]. See [9, 1] for the endpoint cases.

The following is our main result which can be understood as a generalization of the result by Fujita and Tomita [4]. Taking $r_i = 2$ for all $i = 1, \dots, N$ in (1.3), we have (1.2). Si [12] obtained some weighted estimates for multilinear Fourier multipliers with the L^r-based Sobolev regularity, $1 < r \le 2$.

Theorem 1.1. *Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$, $\vec{r} = (r_1, \dots, r_N) \in (1, 2]^N$, $r_N \le r_{N-1} \le \cdots \le r_2 \le r_1$, $\vec{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$ and $n/r_i < s_i \le n$ for all $i = 1, \dots, N$. Assume*

$$p_i > n/s_i \quad \text{and} \quad w_i \in A_{p_i s_i/n} \quad \text{for all} \quad i = 1, \dots, N.$$

Then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)}, \tag{1.3}$$

where $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$ and $H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$ is the Sobolev space of product type with mixed norm which will be defined in Section 2.

2 Preliminaries

2.1 Notations

An operator T acting on N-tuples of functions defined on $\mathbb{R}^n$ is called the N-linear operator. For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \le CB$ for some unspecified constant $C > 0$ independent of A and B, and the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. If $x \in \mathbb{R}^d$, we denotes $(1 + |x|^2)^{1/2}$ by $\langle x\rangle$. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi$$

(See, for example, [3, Chapter 1]). To distinguish linear and multilinear operators, for $m \in L^\infty(\mathbb{R}^n)$, we denote the linear Fourier multiplier operator by $m(D)$ defined by

$$m(D)f(x) = \mathcal{F}^{-1}\left[m(\xi)\widehat{f}(\xi)\right](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi)\widehat{f}(\xi)\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $x, \xi \in \mathbb{R}^n$. Let $0 < p < \infty$ and $w \ge 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \|f\|_{L^p(\mathbb{R}^n,\, w(x)\, dx)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

Let $1 < p < \infty$. We say that a weight w belongs to the Muckenhoupt class A_p if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\, dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B, and p' is the conjugate exponent of p, that is, $1/p + 1/p' = 1$. It is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$([3, Theorem 7.3]).

2.2 Function spaces

To distinguish spaces of usual type and mixed type concerning integrable indices, we use $\mathbb{R}^{Nn}$ and $(\mathbb{R}^n)^N$, respectively.

We recall the definition of L^p-spaces with mixed norm ([2]). Let $\vec{p} = (p_1, \dots, p_N) \in (0, \infty)^N$. The Lebesgue space with mixed norm $L^{\vec{p}}((\mathbb{R}^n)^N)$ consists of all measurable functions F on $\mathbb{R}^{Nn}$ such that

$$\|F\|_{L^{\vec{p}}((\mathbb{R}^n)^N)} = \left\|\|F(x_1, \dots, x_N)\|_{L^{p_1}(\mathbb{R}^n,\, dx_1)} \cdots\right\|_{L^{p_N}(\mathbb{R}^n,\, dx_N)} < \infty,$$

where $(x_1, \cdots, x_N) \in (\mathbb{R}^n)^N$ and dx_i is the Lebesgue measure with respect to the variable x_i for all $i = 1, \dots, N$. In particular, if each p_i is equal to $p \in (0, \infty)$, then we have $\|F\|_{L^{\vec{p}}((\mathbb{R}^n)^N)} = \|F\|_{L^p(\mathbb{R}^{Nn})}$. For $\vec{r} = (r_1, \dots, r_N) \in (1, \infty)^N$ and $\vec{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$, the norm of the Sobolev space of product type with mixed norm $H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is defined by

$$\|F\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)} = \left\|\mathcal{F}^{-1}\left[\langle\xi_1\rangle^{s_1} \cdots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \dots, \xi_N)\right]\right\|_{L^{\vec{r}}((\mathbb{R}^n)^N)},$$

where $\langle\xi_i\rangle = (1 + |\xi_i|^2)^{1/2}$ for $i = 1, \dots, N$ and $\mathcal{F}^{-1}$ is the inverse Fourier transform of $\mathbb{R}^{Nn}$. Taking $r_i = 2$ for all $i = 1, \cdots, N$, we obtain the L^2-based Sobolev space of product type $H^2_{\vec{s}}(\mathbb{R}^{Nn})$, namely, $\|F\|_{H^2_{\vec{s}}(\mathbb{R}^{Nn})} = \|\langle\xi_1\rangle^{s_1} \cdots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \dots, \xi_N)\|_{L^2(\mathbb{R}^{Nn})}$. It should be remarked that if $s_i = s/N, s \ge 0$ for all $i = 1, \cdots, N$,

$$H^2_s(\mathbb{R}^{Nn}) \hookrightarrow H^2_{\vec{s}}(\mathbb{R}^{Nn}),$$

where $H^2_s(\mathbb{R}^{Nn})$ is the L^2-based Sobolev space of usual type, that is to say, $\|F\|_{H^2_s(\mathbb{R}^{Nn})} = \|\langle\xi\rangle^s \widehat{F}\|_{L^2(\mathbb{R}^{Nn})}$, where $\xi \in \mathbb{R}^{Nn}$.

For $\vec{p} = (p_1, \dots, p_N) \in [1, \infty)^N$ and $\vec{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$, the norm of the weighted Lebesgue space with mixed norm $L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is also defined by

$$\|F\|_{L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)} = \left\|\|F(x_1, \dots, x_N)\|_{L^{p_1}(\mathbb{R}^n,\, \langle x_1\rangle^{s_1} dx_1)} \cdots\right\|_{L^{p_N}(\mathbb{R}^n,\, \langle x_N\rangle^{s_N} dx_N)},$$

where $(x_1, \dots, x_N) \in (\mathbb{R}^n)^N$ and $\langle x_i\rangle^{s_i} = (1 + |x_i|^2)^{s_i/2}$ for all $i = 1, \dots, N$. For accuracy, we will frequently write $L^{(p_1, \cdots, p_N)}_{(s_1, \cdots, s_N)}((\mathbb{R}^n)^N)$ instead of $L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)$ in the proof.

For $\vec{p} = (p_1, \dots, p_N)$, $\vec{q} = (q_1, \dots, q_N) \in (0, \infty)^N$, we shall agree that if $a \sim b$ is a relation between numbers a and b, then $\vec{p} \sim \vec{q}$ means that $p_i \sim q_i$ holds for each i.

3 Lemmas

In this section, we give lemmas which play important roles in the proof of Theorem 1.1. The proof of the following lemma is based on the argument of [14, Proposition 1.3.2] or [13, Lemma 3.3].

Lemma 3.1. *Let $r > 0$, $\vec{p} = (p_1, \cdots, p_N)$, $\vec{q} = (q_1, \cdots, q_N) \in [1, \infty)^N$, $\vec{s} = (s_1, \cdots, s_N) \in (\mathbb{R}_{\ge 0})^N$ and $\vec{p} \le \vec{q}$. Then, the estimate*

$$\left\|\langle\xi_1\rangle^{s_1} \dots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \dots, \xi_N)\right\|_{L^{\vec{q}}((\mathbb{R}^n)^N)} \lesssim \left\|\langle\xi_1\rangle^{s_1} \dots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \dots, \xi_N)\right\|_{L^{\vec{p}}((\mathbb{R}^n)^N)}$$

holds, where $\operatorname{supp} F \subset \{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$.

The following is a key lemma in the proof of Theorem 1.1. Fujita and Tomita [4, Proposition A.2] proved (1.2) by using the fact that $H^2_{\vec{s}}(\mathbb{R}^{Nn})$ is a multiplication algebra when $s_i > n/2$ for all $i = 1, \cdots, N$. Instead of this, we shall use the following lemma.

Lemma 3.2. *Let $N_0 \in \mathbb{N}$, $\vec{r} = (r_1, \dots, r_N) \in (1, 2]^N$, $r_N \le r_{N-1} \le \cdots \le r_2 \le r_1$, $\vec{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$, $n/r_i < s_i \le n$ and $n/s_i < q_i < r_i$ for all $i = 1, \dots, N$. Then, the estimate*

$$\left\|\mathcal{F}\left[m(2^j\cdot)\Psi(\cdot/2^k)\right]\right\|_{L^{(q_1', \dots, q_N')}_{(s_1 q_1', \dots, s_N q_N')}((\mathbb{R}^n)^N)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)}$$

holds for all $j \in \mathbb{Z}$, $-N_0 \le k \le N_0$ and $m \in H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$.

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